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Prolongation structure of the (2+1)-dimensional integrable Heisenberg ferromagnet model

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Abstract

We investigate the integrable (2+1)-dimensional (modified) Heisenberg ferromagnet (HF) model using the prolongation structure theory. For the integrable (2+1)-dimensional modified HF models, the corresponding geometrical equivalent counterparts, such as the (2+1)-dimensional nonlinear Schrödinger equation and the coupled (2+1)-dimensional integrable equations, are also presented through the motion of Minkowski space curves endowed with an additional spatial variable.

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1. Introduction

The prolongation structure theory proposed by Wahlquist and Estabrook [1] is a very useful and effective tool in the analysis of (1+1)-dimensional integrable systems. On the basis of this theory, the (1+1)-dimensional integrable (modified) HF models have been investigated in [2, 3]. Moreover, in terms of the motion of curves in Euclidean and Minkowski space, their geometrical equivalent counterparts have also been given there, such as focusing and defocusing nonlinear Schrödinger equations $NLS^{\pm}$ and the coupled integrable equation derived by Nakayama [4].

With the successful application of Wahlquist and Estabrook's prolongation structure theory, a prolongation structure method to discuss an evolution equation in two spatial dimensions was proposed by Morris [5], but its further applications to the (2+1)-dimensional integrable systems are fewer in number. The reason lies in the fact that equations with more independent variables do not possess nontrivial finite-dimensional coverings [6]. The Heisenberg ferromagnet (HF) model is an important integrable system. Many efforts have been devoted to the study of its (2+1)-dimensional extensions [7, 8]. The question that naturally arises is whether the prolongation structure method can be used to investigate the (2+1)-dimensional (modified) HF models. The purpose of this paper is to give affirmative answer to this question. Note that the spectral parameter in the Lax representation of (2+1)-dimensional integrable HF model can be dependent on the time and space variables. Therefore, in this paper, we shall consider the case of general prolongations of Morris's theory and apply it to analyse the (2+1)-dimensional integrable (modified) HF model.

2. The (2+1)-dimensional integrable Heisenberg ferromagnet model

The well-known (1+1)-dimensional integrable HF equation is given by

$$\mathbf{S_t} = \mathbf{S} \times \mathbf{S}_{xx}, \tag{1}$$

where the subscripts stand for partial derivatives, $\mathbf{S} = (\mathbf{S_1}, \mathbf{S_2}, \mathbf{S_3})$ is the spin vector with $\mathbf{S^2} = \mathbf{1}$ and '$\times$' denotes the cross product. Its (2+1)-dimensional integrable extensions have drawn wide interest. A simple (2+1)-dimensional integrable HF equation is given by [8]

$$\mathbf{S_t} = \{\mathbf{S} \times \mathbf{S_y} + u\mathbf{S}\}_x, \qquad u_x = -\mathbf{S} \cdot (\mathbf{S_x} \times \mathbf{S_y}). \tag{2}$$

We now analyse this equation by using the prolongation structure theory. Let us first consider the case for $\mathbf{S}_t = 0$ in (2). Setting $\mathbf{W} = \mathbf{S_x}$, $\mathbf{T} = \mathbf{S_y}$ and taking $\mathbf{S}$, $\mathbf{T}$, $\mathbf{W}$ and u as the new independent variables, we can define the following set of two-forms,

$$\begin{aligned}
&\alpha_a = \mathrm{d}S_a \wedge \mathrm{d}x - T_a\,\mathrm{d}y \wedge \mathrm{d}x, \qquad \alpha_{a+3} = \mathrm{d}S_a \wedge \mathrm{d}y - W_a\,\mathrm{d}x \wedge \mathrm{d}y,\\
&\alpha_{a+6} = (\mathbf{W} \times \mathbf{T})_a\,\mathrm{d}x \wedge \mathrm{d}y + (\mathbf{S} \times \mathrm{d}\mathbf{T})_a \wedge \mathrm{d}y + S_a\,\mathrm{d}u \wedge \mathrm{d}y + uW_a\,\mathrm{d}x \wedge \mathrm{d}y,\\
&\alpha_{10} = \mathrm{d}u \wedge \mathrm{d}y + \mathbf{S} \cdot (\mathbf{W} \times \mathbf{T})\,\mathrm{d}x \wedge \mathrm{d}y, \qquad \alpha_{a+10} = \mathrm{d}T_a \wedge \mathrm{d}y + \mathrm{d}W_a \wedge \mathrm{d}x,\\
&\alpha_{14} = (\mathbf{T} \cdot \mathbf{W})\,\mathrm{d}x \wedge \mathrm{d}y + S_a \cdot \mathrm{d}T_a \wedge \mathrm{d}y,
\end{aligned} \tag{3}$$

where $a = 1, 2, 3$, such that they constitute a closed ideal $I = \{\alpha_i, i = 1, 2, \ldots, 14\}$. When these two-forms are null, it leads to (2) in which $\mathbf{S}_t = 0$. In order to establish the prolongation structure, we extend the above ideal I by adding to it a set of one-forms,

$$\Omega^k = \mathrm{d}\xi^k + F^k(x, y, \mathbf{S}, \mathbf{T}, \mathbf{W}, u)\xi^k\,\mathrm{d}x + G^k(x, y, \mathbf{S}, \mathbf{T}, \mathbf{W}, u)\xi^k\,\mathrm{d}y, \qquad k = 1, 2, \ldots, n, \tag{4}$$

where ξ^k is the prolongation variable. The extended ideal must be closed under exterior differentiations, i.e.,

$$\mathrm{d}\Omega^k = \sum_{i=1}^{14} g^{ki}\alpha_i + \sum_{j=1}^{n} \zeta_j^k \wedge \Omega^j, \tag{5}$$

where g^{ki} and ζ_j^k are some sets of 0-forms and 1-forms, respectively. On imposing this requirement, we obtain the following set of partial differential equations for F^k and G^k,

$$\begin{aligned}
&\frac{\partial F^k}{\partial T_a} = \frac{\partial F^k}{\partial W_a} = \frac{\partial F^k}{\partial u} = 0, \qquad \frac{\partial G^k}{\partial W_a} = 0,\\
&-\frac{\partial F^k}{\partial S_a}T_a + \frac{\partial G^k}{\partial S_a}W_a - \frac{\partial G^k}{\partial u}\mathbf{S} \cdot (\mathbf{W} \times \mathbf{T}) + \frac{\partial G^k}{\partial T_a}\{[\mathbf{S} \times (\mathbf{W} \times \mathbf{T})]_a\\
&\qquad - S_a(\mathbf{T} \cdot \mathbf{W}) + u(\mathbf{S} \times \mathbf{W})_a\} - [F, G]^k + \frac{\partial G^k}{\partial x} - \frac{\partial F^k}{\partial y} = 0,
\end{aligned} \tag{6}$$

where

$$[F, G]^k \equiv \sum_{l=1}^{n} F^l \frac{\partial G^k}{\partial y^l} - \sum_{l=1}^{n} G^l \frac{\partial F^k}{\partial y^l}.$$

By solving (6), we have the following solution,

$$F = \lambda \sum_{i=1}^{3} S_i X_i, \qquad G = u \sum_{i=1}^{3} S_i X_i + \sum_{i=1}^{3} (\mathbf{S} \times \mathbf{T})_i X_i, \qquad \frac{\partial \lambda}{\partial y} = 0, \tag{7}$$

where X_i, $i = 1, 2, 3$, depend only on the prolongation variables ξ^k and have the commutation relation of the $su(2)$ Lie algebra.

Let us now turn to discuss the case for equation (2). We define a set of 3-form $\overline{\alpha}_i$ as follows,

$$\begin{aligned}
&\overline{\alpha}_a = \mathrm{d}S_a \wedge \mathrm{d}x \wedge \mathrm{d}t - T_a\, \mathrm{d}y \wedge \mathrm{d}x \wedge \mathrm{d}t, \qquad \overline{\alpha}_{a+3} = \mathrm{d}S_a \wedge \mathrm{d}y \wedge \mathrm{d}t - W_a\, \mathrm{d}x \wedge \mathrm{d}y \wedge \mathrm{d}t,\\
&\overline{\alpha}_{a+6} = (\mathbf{W} \times \mathbf{T})_a\, \mathrm{d}x \wedge \mathrm{d}y \wedge \mathrm{d}t + (\mathbf{S} \times \mathrm{d}\mathbf{T})_a \wedge \mathrm{d}y \wedge \mathrm{d}t + S_a\, \mathrm{d}u \wedge \mathrm{d}y \wedge \mathrm{d}t\\
&\qquad\qquad + u W_a\, \mathrm{d}x \wedge \mathrm{d}y \wedge \mathrm{d}t - \mathrm{d}S_a \wedge \mathrm{d}x \wedge \mathrm{d}y,\\
&\overline{\alpha}_{10} = \mathrm{d}u \wedge \mathrm{d}y \wedge \mathrm{d}t + \mathbf{S} \cdot (\mathbf{W} \times \mathbf{T})\, \mathrm{d}x \wedge \mathrm{d}y \wedge \mathrm{d}t,\\
&\overline{\alpha}_{a+10} = \mathrm{d}T_a \wedge \mathrm{d}y \wedge \mathrm{d}t + \mathrm{d}W_a \wedge \mathrm{d}x \wedge \mathrm{d}t,\\
&\overline{\alpha}_{14} = (\mathbf{T} \cdot \mathbf{W})\, \mathrm{d}x \wedge \mathrm{d}y \wedge \mathrm{d}t + S_a \cdot \mathrm{d}T_a \wedge \mathrm{d}y \wedge \mathrm{d}t,
\end{aligned} \tag{8}$$

where $a = 1, 2, 3$, such that they constitute a closed ideal. When these two-forms are null, we recover (2). According to the prolongation structure theory of Morris [5], we may introduce the following two-forms,

$$\overline{\Omega}^k = \Omega^k \wedge \mathrm{d}t + H^k_j \xi^j\, \mathrm{d}x \wedge \mathrm{d}y + \left(A^k_j\, \mathrm{d}x + B^k_j\, \mathrm{d}y\right) \wedge \mathrm{d}\xi^j, \qquad k = 1, 2, \ldots, n, \tag{9}$$

Note that the case of the general prolongations is considered here. Therefore, we demand that the matrices of A and B depend on the variables (x, y, t) and the form of Ω^k is given by (4), in which λ depends on the variables (x, y, t) and $\lambda_y \neq 0$ due to the new variable t. It is easily shown that

$$\mathrm{d}\overline{\Omega}^k = \sum_{i=1}^{14} g^{ki} \overline{\alpha}_i + \sum_{j=1}^{n} \zeta^k_j \wedge \overline{\Omega}^j, \tag{10}$$

provided that the matrix H is given by

$$H = GA - FB + A_y - B_x \tag{11}$$

and

$$\begin{aligned}
&\mathrm{d}H \wedge \mathrm{d}x \wedge \mathrm{d}y - \frac{\partial G}{\partial T_a} (\mathbf{S} \times \mathrm{d}\mathbf{S})_a \wedge \mathrm{d}x \wedge \mathrm{d}y - \lambda_y S_a X_a\, \mathrm{d}x \wedge \mathrm{d}y \wedge \mathrm{d}t\\
&\qquad - A_t G\, \mathrm{d}x \wedge \mathrm{d}y \wedge \mathrm{d}t + B_t F\, \mathrm{d}x \wedge \mathrm{d}y \wedge \mathrm{d}t = 0.
\end{aligned} \tag{12}$$

Substituting expressions (7) of F and G into (11) and (12), we obtain

$$A = 0, \qquad B = \frac{1}{\lambda} I, \tag{13}$$

and

$$\lambda_t = -\lambda \lambda_y, \qquad \lambda_x = 0. \tag{14}$$

By restricting (9) on the solution manifold, we obtain the Lax representation of the (2+1)-dimensional integrable HF equation ({2})

$$\begin{aligned}
\xi_x &= -F|_{X_i = \frac{\mathrm{i}}{2}\sigma_i} \xi = -\frac{\mathrm{i}\lambda}{2} \sum_{i=1}^{3} S_i \sigma_i \xi,\\
\xi_t &= -\frac{1}{B} \xi_y - \frac{1}{B} G|_{X_i = \frac{\mathrm{i}}{2}\sigma_i} \xi\\
&= -\lambda \xi_y - \frac{\mathrm{i}\lambda}{2} \sum_{i=1}^{3} [u S_i \sigma_i + (\mathbf{S} \times \mathbf{T})_i \sigma_i] \xi,
\end{aligned} \tag{15}$$

where $\sigma_i, i = 1, 2, 3$, are Pauli matrices, and the spectral parameter satisfies the nonlinear equation (14).

The space curve formalism plays an important role in the understanding of the nonlinear integrable equations. By associating with the motion of Euclidean space curves endowed with an extra spatial variable, Lakshmanan *et al* showed that the (2+1)-dimensional extension of HF model (2) is actually geometric equivalent to the (2+1)-dimensional NLS^+ [9],

$$\mathrm{i}\psi_t - \psi_{xy} - R\psi = 0, \qquad R_x = \tfrac{1}{2}\partial_y|\psi|^2. \tag{16}$$

Its Lax representation is given by

$$\Phi_x = U\Phi, \qquad \Phi_t = V\Phi + \lambda\Phi_y, \tag{17}$$

in which

$$U = \begin{pmatrix} \mathrm{i}\lambda/2 & \psi/2 \\ -\psi^*/2 & -\mathrm{i}\lambda/2 \end{pmatrix}, \qquad V = \begin{pmatrix} -\frac{\mathrm{i}}{2}R & -\frac{\mathrm{i}}{2}\psi_y \\ -\frac{\mathrm{i}}{2}\psi_y^* & \frac{\mathrm{i}}{2}R \end{pmatrix}, \tag{18}$$

and the spectral parameter satisfies (14).

3. The (2+1)-dimensional integrable modified Heisenberg ferromagnet model

The (1+1)-dimensional integrable modified HF equation is given by

$$\mathbf{S}_t = \mathbf{S}\,\bar{\times}\,\mathbf{S}_{xx}, \tag{19}$$

where $\mathbf{S} = (S_1, S_2, S_3)$ with $\mathbf{S}^2 \equiv \mathbf{S}\circ\mathbf{S} \equiv S_1^2 + S_2^2 - S_3^2 = \pm 1$ and $S_3 > 0$, and $\bar{\times}$ denotes the pseudo-cross product, i.e., $\mathbf{S}\,\bar{\times}\,\mathbf{S}_{xx} = \left(S_2S_{3_{xx}} - S_3S_{2_{xx}}, S_3S_{1_{xx}} - S_1S_{3_{xx}}, -S_1S_{2_{xx}} + S_2S_{1_{xx}}\right)$. In this section, using the prolongation structure method as done in the previous section, we shall analyse the (2+1)-dimensional extensions of (19) and give their corresponding geometrical equivalent counterparts.

A (2+1)-dimensional integrable modified HF model which is gauge equivalent to the (2+1)-dimensional NLS^- is given by [10]

$$\mathbf{S_t} = \{\mathbf{S}\,\bar{\times}\,\mathbf{S_y} + u\mathbf{S}\}_x, \qquad u_x = \mathbf{S}\circ(\mathbf{S_x}\,\bar{\times}\,\mathbf{S_y}), \tag{20}$$

where $\mathbf{S}\circ\mathbf{S} = S_1^2 + S_2^2 - S_3^2 = -1$.

As in the previous section, setting $\mathbf{W} = \mathbf{S_x}$, $\mathbf{T} = \mathbf{S_y}$ and taking $\mathbf{S}$, $\mathbf{T}$, $\mathbf{W}$ and u as the new independent variables, we can define a set of two-forms

$$\begin{aligned}
\tilde{\alpha}_a &= \mathrm{d}S_a \wedge \mathrm{d}x - T_a\,\mathrm{d}y \wedge \mathrm{d}x,\\
\tilde{\alpha}_{a+3} &= \mathrm{d}S_a \wedge \mathrm{d}y - W_a\,\mathrm{d}x \wedge \mathrm{d}y,\\
\tilde{\alpha}_{a+6} &= (\mathbf{W}\,\bar{\times}\,\mathbf{T})_a\,\mathrm{d}x \wedge \mathrm{d}y + (\mathbf{S}\,\bar{\times}\,\mathrm{d}\mathbf{T})_a \wedge \mathrm{d}y + S_a\,\mathrm{d}u \wedge \mathrm{d}y + uW_a\,\mathrm{d}x \wedge \mathrm{d}y,\\
\tilde{\alpha}_{10} &= \mathrm{d}u \wedge \mathrm{d}y - \mathbf{S}\circ(\mathbf{W}\,\bar{\times}\,\mathbf{T})\,\mathrm{d}x \wedge \mathrm{d}y,\\
\tilde{\alpha}_{a+10} &= \mathrm{d}T_a \wedge \mathrm{d}y + \mathrm{d}W_a \wedge \mathrm{d}x,\\
\tilde{\alpha}_{14} &= (\mathbf{T}\circ\mathbf{W})\,\mathrm{d}x \wedge \mathrm{d}y + \mathbf{S}\circ\mathrm{d}\mathbf{T} \wedge \mathrm{d}y,
\end{aligned} \tag{21}$$

where $a = 1, 2, 3$, such that they constitute a closed ideal $\tilde{I} = \{\tilde{\alpha}_i, i = 1, 2, \ldots, 14\}$. When the above two-forms are null, we recover (20) for the case of $\mathbf{S}_t = 0$. Then we extend the above ideal $\tilde{I}$ by adding to it a set of of one-forms,

$$\tilde{\Omega}^k = \mathrm{d}\xi + \hat{F}^k(x, y, \mathbf{S}, \mathbf{T}, \mathbf{W}, u)\xi\,\mathrm{d}x + \hat{G}^k(x, y, \mathbf{S}, \mathbf{T}, \mathbf{W}, u)\xi\,\mathrm{d}y, \qquad k = 1, 2, \ldots, n, \tag{22}$$

where ξ^k is prolongation variable. By requiring the ideal $\hat{I} = \{\tilde{\alpha}_a, \tilde{\Omega}^k\}$ to be closed under exterior differentiation, we obtain the following set of partial differential equations for $\hat{F}^k$ and $\hat{G}^k$,

$$\frac{\partial \hat{F}^k}{\partial T_a} = \frac{\partial \hat{F}^k}{\partial W_a} = \frac{\partial \hat{F}^k}{\partial u} = 0, \qquad \frac{\partial \hat{G}^k}{\partial W_a} = 0,$$
$$-\frac{\partial \hat{F}^k}{\partial S_a} T_a + \frac{\partial \hat{G}^k}{\partial S_a} W_a + \frac{\partial \hat{G}^k}{\partial u} \mathbf{S} \circ (\mathbf{W} \bar{\times} \mathbf{T}) + \frac{\partial \hat{G}^k}{\partial T_a} \{[\mathbf{S} \bar{\times} (\mathbf{W} \bar{\times} \mathbf{T})]_a + S_a (\mathbf{T} \circ \mathbf{W})$$
$$+ u (\mathbf{S} \bar{\times} \mathbf{W})_a\} - [\hat{F}, \hat{G}]^k + \frac{\partial \hat{G}^k}{\partial x} - \frac{\partial \hat{F}^k}{\partial y} = 0, \tag{23}$$

where

$$[\hat{F}, \hat{G}]^k = \sum_{l=1}^{n} \hat{F}^l \frac{\partial \hat{G}^k}{\partial y^l} - \sum_{l=1}^{n} \hat{G}^l \frac{\partial \hat{F}^k}{\partial y^l}.$$

By solving (23), we obtain

$$\hat{F} = \lambda \sum_{i=1}^{3} S_i \hat{X}_i, \qquad \hat{G} = u \sum_{i=1}^{3} S_i \hat{X}_i + \sum_{i=1}^{3} (\mathbf{S} \bar{\times} \mathbf{T})_i \hat{X}_i, \qquad \frac{\partial \lambda}{\partial y} = 0, \tag{24}$$

where $\hat{X}_i$, $i = 1, 2, 3$, depend only on the prolongation variables ξ^k and have the commutation relation of the $su(1, 1)$ Lie algebra.

Similar to that of the (2+1)-dimensional integrable HF equation, we now define the following set of 3-form $\hat{\alpha}_i$

$$\hat{\alpha}_a = \mathrm{d}S_a \wedge \mathrm{d}x \wedge \mathrm{d}t - T_a\, \mathrm{d}y \wedge \mathrm{d}x \wedge \mathrm{d}t,$$
$$\hat{\alpha}_{a+3} = \mathrm{d}S_a \wedge \mathrm{d}y \wedge \mathrm{d}t - W_a\, \mathrm{d}x \wedge \mathrm{d}y \wedge \mathrm{d}t,$$
$$\hat{\alpha}_{a+6} = (\mathbf{W} \bar{\times} \mathbf{T})_a\, \mathrm{d}x \wedge \mathrm{d}y \wedge \mathrm{d}t + (\mathbf{S} \bar{\times} \mathrm{d}\mathbf{T})_a \wedge \mathrm{d}y \wedge \mathrm{d}t + S_a\, \mathrm{d}u \wedge \mathrm{d}y \wedge \mathrm{d}t$$
$$+ u W_a\, \mathrm{d}x \wedge \mathrm{d}y \wedge \mathrm{d}t - \mathrm{d}S_a \wedge \mathrm{d}x \wedge \mathrm{d}y, \tag{25}$$
$$\hat{\alpha}_{10} = \mathrm{d}u \wedge \mathrm{d}y \wedge \mathrm{d}t - \mathbf{S} \circ (\mathbf{W} \bar{\times} \mathbf{T})\, \mathrm{d}x \wedge \mathrm{d}y \wedge \mathrm{d}t,$$
$$\hat{\alpha}_{a+10} = \mathrm{d}T_a \wedge \mathrm{d}y \wedge \mathrm{d}t + \mathrm{d}W_a \wedge \mathrm{d}x \wedge \mathrm{d}t,$$
$$\hat{\alpha}_{14} = (\mathbf{T} \circ \mathbf{W})\, \mathrm{d}x \wedge \mathrm{d}y \wedge \mathrm{d}t + \mathbf{S} \circ \mathrm{d}\mathbf{T} \wedge \mathrm{d}y \wedge \mathrm{d}t,$$

where $a = 1, 2, 3$, such that they constitute a closed ideal. Then we introduce the following two-forms,

$$\hat{\Omega}^k = \tilde{\Omega}^k \wedge \mathrm{d}t + \hat{H}^k_j \xi^j\, \mathrm{d}x \wedge \mathrm{d}y + \left(\hat{A}^k_j\, \mathrm{d}x + \hat{B}^k_j\, \mathrm{d}y\right) \wedge \mathrm{d}\xi^j, \qquad k = 1, 2, \ldots, n, \tag{26}$$

where the matrices of $\hat{A}$ and $\hat{B}$ depend on the variables (x, y, t) and the form of $\tilde{\Omega}^k$ is given by (22), in which λ depending on the variables (x, y, t) and $\lambda_y \neq 0$. It can be shown that

$$\mathrm{d}\hat{\Omega}^k = \sum_{i=1}^{14} f^{ki} \hat{\alpha}_i + \sum_{j=1}^{n} \eta^k_j \wedge \hat{\Omega}^j, \tag{27}$$

provided that the matrix $\hat{H}$ is given by

$$\hat{H} = \hat{G}\hat{A} - \hat{F}\hat{B} + \hat{A}_y - \hat{B}_x, \tag{28}$$

and

$$\mathrm{d}\hat{H} \wedge \mathrm{d}x \wedge \mathrm{d}y - \frac{\partial \hat{G}}{\partial T_a} (\mathbf{S} \times \mathrm{d}\mathbf{S})_a \wedge \mathrm{d}x \wedge \mathrm{d}y - \lambda_y S_a X_a\, \mathrm{d}x \wedge \mathrm{d}y \wedge \mathrm{d}t$$
$$- \hat{A}_t \hat{G}\, \mathrm{d}x \wedge \mathrm{d}y \wedge \mathrm{d}t + \hat{B}_t \hat{F}\, \mathrm{d}x \wedge \mathrm{d}y \wedge \mathrm{d}t = 0. \tag{29}$$

Substituting expressions (24) of $\hat{F}$ and $\hat{G}$ into (28) and (29), we obtain

$$\hat{A} = 0, \qquad \hat{B} = \frac{1}{\lambda} I, \tag{30}$$

and the nonlinear equation (14) for the spectral parameter λ.

By restricting (26) on the solution manifold, we obtain the Lax representation of (20)

$$\begin{aligned} \xi_x &= -\hat{F}|_{X_i=\tau_i}\xi = -\lambda \sum_{i=1}^{3} S_i \tau_i \xi, \\ \xi_t &= -\frac{1}{B}\xi_y - \frac{1}{B} G|_{X_i=\tau_i}\xi \\ &= -\lambda \xi_y - \lambda \sum_{i=1}^{3} [u S_i \tau_i + (\mathbf{S} \bar{\times} \mathbf{T})_i \tau_i]\xi. \end{aligned} \tag{31}$$

where τ_i are the generators of Lie algebra $su(1,1)$, i.e., $\tau_1 = \frac{1}{2}\begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}$, $\tau_2 = \frac{1}{2}\begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix}$, $\tau_3 = \frac{1}{2}\begin{pmatrix} 0 & \mathrm{i} \\ \mathrm{i} & 0 \end{pmatrix}$.

In order to give the geometrical equivalent counterpart of (20), we identify $\mathbf{S}$ with the tangent of a Minkowski space curve and endow the moving curve with an additional spatial variable y. Thus equation (20) can be rewritten as

$$\mathbf{t}_t = \mathbf{t}_x \bar{\times} \mathbf{t}_y + \mathbf{t} \bar{\times} \mathbf{t}_{yx} + u_x \mathbf{t} + u \mathbf{t}_x, \tag{32}$$

where the subscript x denotes the arc length parameter. On imposing the compatibility condition $\mathbf{t}_{xy} = \mathbf{t}_{yx}$, $\mathbf{n}_{xy} = \mathbf{n}_{yx}$, $\mathbf{b}_{xy} = \mathbf{b}_{yx}$, we obtain the following y-part equations of the orthogonal trihedral,

$$\begin{aligned} &\mathbf{t}_y = \frac{u_x}{\kappa}\mathbf{b} + \partial_x^{-1}\left(\kappa_y + \frac{\tau u_x}{\kappa}\right)\mathbf{n}, \qquad \mathbf{n}_y = \left(u + \partial_x^{-1}\tau_y\right)\mathbf{b} + \partial_x^{-1}\left(\kappa_y + \frac{\tau u_x}{\kappa}\right)\mathbf{t}, \\ &\mathbf{b}_y = -\left(u + \partial_x^{-1}\tau_y\right)\mathbf{n} + \frac{u_x}{\kappa}\mathbf{t}. \end{aligned} \tag{33}$$

Substituting (33) and (A.1) into (32), we obtain

$$\mathbf{t}_t = \tfrac{1}{2}(-\mathrm{i}\psi_y \mathbf{N}^* + \mathrm{i}\psi_y^* \mathbf{N}). \tag{34}$$

Comparing (34) with (A.7), we have

$$\gamma = -\mathrm{i}\psi_y. \tag{35}$$

Substituting (35) into (A.8), we obtain the well-known (2+1)-dimensional NLS^-

$$\mathrm{i}\psi_t - \psi_{xy} + R\psi = 0, \qquad R_x = \tfrac{1}{2}\partial_y |\psi|^2. \tag{36}$$

Its Lax representation is given by

$$\Phi_x = U\Phi, \qquad \Phi_t = V\Phi + \lambda\Phi_y, \tag{37}$$

in which

$$U = \begin{pmatrix} \mathrm{i}\lambda/2 & \psi/2 \\ \psi^*/2 & -\mathrm{i}\lambda/2 \end{pmatrix}, \qquad V = \begin{pmatrix} -\frac{\mathrm{i}}{2}R & -\frac{\mathrm{i}}{2}\psi_y \\ \frac{\mathrm{i}}{2}\psi_y^* & \frac{\mathrm{i}}{2}R \end{pmatrix}. \tag{38}$$

and the spectral parameter satisfies (14).

Proceeding with the similar procedures as done to (20), we may get the following (2+1)-dimensional integrable modified HF equation,

$$\mathbf{S}_t = \{\mathbf{S} \bar{\times} \mathbf{S}_y + u\mathbf{S}\}_x, \qquad u_x = -\mathbf{S} \circ (\mathbf{S}_x \bar{\times} \mathbf{S}_y), \tag{39}$$

where $\mathbf{S} \circ \mathbf{S} = 1$. The corresponding Lax representation is given by

$$\xi_x = -\lambda \sum_{i=1}^{3} S_i \tau_i \xi, \qquad \xi_t = -\lambda \xi_y - \lambda \sum_{i=1}^{3} [u S_i \tau_i + (\mathbf{S} \bar{\times} \mathbf{T})_i \tau_i] \xi, \tag{40}$$

and the spectral parameter satisfies

$$\lambda_t = \lambda \lambda_y, \qquad \lambda_x = 0. \tag{41}$$

We now associate the (2+1)-dimensional modified HF equation (39) with a moving Minkowski space curve parametrized by the arc length x and endowed with an additional coordinate y. Let us first consider the case: $\mathbf{S} = \mathbf{b}$. In this case equation (39) can be rewritten as

$$\mathbf{b}_t = \mathbf{b}_x \bar{\times} \mathbf{b}_y + \mathbf{b} \bar{\times} \mathbf{b}_{yx} + u_x \mathbf{b} + u \mathbf{b}_x. \tag{42}$$

By requiring the compatibility conditions $\mathbf{t}_{xy} = \mathbf{t}_{yx}$, $\mathbf{n}_{xy} = \mathbf{n}_{yx}$ and $\mathbf{b}_{xy} = \mathbf{b}_{yx}$, we obtain the y-part equations of the orthogonal trihedral,

$$\begin{aligned} &\mathbf{t}_y = -\frac{u_x}{\tau}\mathbf{b} + \partial_x^{-1}(\kappa_y - u_x)\mathbf{n}, \qquad \mathbf{n}_y = \partial_x^{-1}\left(\tau_y - \frac{\kappa u_x}{\tau}\right)\mathbf{b} + \partial_x^{-1}(\kappa_y - u_x)\mathbf{t}, \\ &\mathbf{b}_y = -\partial_x^{-1}\left(\tau_y - \frac{\kappa u_x}{\tau}\right)\mathbf{n} - \frac{u_x}{\tau}\mathbf{t}. \end{aligned} \tag{43}$$

Substituting (43) and (A.1) into (42) and comparing it with (A.10), we obtain

$$\hat{\zeta} = -\hat{\phi}_y, \qquad \hat{\beta} = \hat{\psi}_y. \tag{44}$$

Then from (44) and (A.11), we obtain the following integral equation for $\hat{\psi}$ and $\hat{\phi}$,

$$\hat{\psi}_t + \hat{\psi}_{xy} - \hat{\gamma}\hat{\psi} = 0, \qquad \hat{\phi}_t - \hat{\phi}_{xy} + \hat{\gamma}\hat{\phi} = 0, \qquad \hat{\gamma}_x = -\partial_y(\hat{\phi}\hat{\psi}). \tag{45}$$

Note that if the reduction $\partial_y = \partial_x$ is imposed, then equation (45) reduces to the (1+1)-dimensional coupled integrable equation derived by Nakayama [4]. The Lax representation of (45) is given by

$$\Phi_x = U\Phi, \qquad \Phi_t = V\Phi + \mathrm{i}\lambda\Phi_y, \tag{46}$$

in which

$$U = \begin{pmatrix} \mathrm{i}\lambda/2 & \hat{\phi}/\sqrt{2} \\ -\hat{\psi}/\sqrt{2} & -\mathrm{i}\lambda/2 \end{pmatrix}, \qquad V = \begin{pmatrix} -\frac{\hat{\gamma}}{2} & \frac{\hat{\phi}_y}{\sqrt{2}} \\ \frac{\hat{\psi}_y}{\sqrt{2}} & \frac{\hat{\gamma}}{2} \end{pmatrix}, \tag{47}$$

and the spectral parameter satisfies (41).

For the other case, i.e., $\mathbf{S} = \mathbf{n}$, we may rewrite (39) as

$$\mathbf{n}_t = \mathbf{n}_x \bar{\times} \mathbf{n}_y + \mathbf{n} \bar{\times} \mathbf{n}_{yx} + u_x \mathbf{n} + u \mathbf{n}_x. \tag{48}$$

The compatibility conditions, $\mathbf{t}_{xy} = \mathbf{t}_{yx}$, $\mathbf{n}_{xy} = \mathbf{n}_{yx}$ and $\mathbf{b}_{xy} = \mathbf{b}_{yx}$, give the following y-part equations of the orthogonal trihedral,

$$\begin{aligned} &\mathbf{t}_y = u\mathbf{b} + \partial_x^{-1}(\kappa_y + \tau u)\mathbf{n}, \qquad \mathbf{b}_y = u\mathbf{t} - \partial_x^{-1}(\kappa u + \tau_y)\mathbf{n}, \\ &\mathbf{n}_y = \partial_x^{-1}(\kappa_y + \tau u)\mathbf{t} + \partial_x^{-1}(\kappa u + \tau_y)\mathbf{b}. \end{aligned} \tag{49}$$

Substituting (49) and (A.1) into (48) and comparing it with (A.13), we obtain

$$\tilde{\zeta} = -\tilde{\psi}_y, \qquad \tilde{\beta} = -\tilde{\phi}_y. \tag{50}$$

Then from (50) and (A.14), we obtain the following integral equations:

$$\tilde{\psi}_t + \tilde{\psi}_{xy} + \tilde{\psi}\tilde{\gamma} = 0, \qquad \tilde{\phi}_t - \tilde{\phi}_{xy} - \tilde{\phi}\tilde{\gamma} = 0, \qquad \tilde{\gamma}_x = -\partial_y(\tilde{\phi}\tilde{\psi}). \tag{51}$$

Under the reduction $\partial_y = \partial_x$, equation (51) reduces to another (1+1)-dimensional coupled integrable equation derived by Nakayama [4]. The Lax representation of (51) is given by (46), in which U and V are given by

$$U = \begin{pmatrix} \mathrm{i}\lambda/2 & -\tilde{\phi}/\sqrt{2} \\ -\tilde{\psi}/\sqrt{2} & -\mathrm{i}\lambda/2 \end{pmatrix}, \qquad V = \begin{pmatrix} \frac{\tilde{\gamma}}{2} & -\frac{\tilde{\phi}_y}{\sqrt{2}} \\ \frac{\tilde{\psi}_y}{\sqrt{2}} & -\frac{\tilde{\gamma}}{2} \end{pmatrix}, \tag{52}$$

and the spectral parameter satisfies (41).

To summarize, we have investigated the integrable (2+1)-dimensional (modified) HF model by using the Morris's prolongation structure theory. Note that the spectral parameter in the Lax representation of (2+1)-dimensional integrable HF model can be dependent on the time and space variables. Therefore, the case of general prolongations of Morris's theory is considered in this paper. For the integrable (2+1)-dimensional modified HF models, through the motion of Minkowski space curves endowed with an additional spatial variable, we have presented the corresponding geometric equivalent (2+1)-dimensional integrable equations, especially the (2+1)-dimensional integrable extensions of the (1+1)-dimensional coupled integrable equations derived by Nakayama [4] which, to our knowledge, have not been reported so far. It should be pointed out that our approach can be used to analyse more complex (2+1)-dimensional extensions of the (modified) HF model.

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Appendix. The motion of a curve in Minkowski space

The Minkowski space M^3 is defined as a space to be the usual three-dimensional R-vector space consisting of vectors $\{(x_1, x_2, x_3)|x_1, x_2, x_3 \in R\}$, but endowed with the inner product, $X \circ Y = x_1y_1 + x_2y_2 - x_3y_3$. The vector product of two vectors A and B can be defined by requiring the relation, $(A \bar{\times} B) \circ C = \mathrm{Det}(A, B, C)$ for all C. Now we can define 3-frames as follows. For two vectors $\mathbf{e_1}$ and $\mathbf{e_2}$, for which $\mathbf{e}_i \circ \mathbf{e}_i = \pm 1$ and $\mathbf{e}_1 \circ \mathbf{e}_2 = 0$, a third is defined by $\mathbf{e}_3 = \mathbf{e}_1 \bar{\times} \mathbf{e}_2$, and these three vector form an orthonormal 3-frame. If we define $\epsilon, \eta \in \{1, -1\}$ by $\mathbf{e}_1 \circ \mathbf{e}_1 = \epsilon, \mathbf{e}_2 \circ \mathbf{e}_2 = \eta$, then it follows that $\mathbf{e}_3 \circ \mathbf{e}_3 = -\epsilon\eta$. In this paper, we take $\epsilon = -1$ and $\eta = 1$. Let c be a curve in M^3, which we assume is parametrized by arc length s and satisfies $c'' \circ c'' \neq 0$. Then this curve induces a Frenet–Serret 3-frame $\mathbf{t} = \mathbf{e}_1 = c'$, $\mathbf{n} = \mathbf{e}_2 = c''/\sqrt{|c'' \circ c''|}$, $\mathbf{b} = \mathbf{e}_3 = \mathbf{e}_1 \bar{\times} \mathbf{e}_2$, for which the following Frenet–Serret equations hold,

$$\mathbf{t}_s = k\mathbf{n}, \qquad \mathbf{n}_s = k\mathbf{t} + \tau\mathbf{b}, \qquad \mathbf{b}_s = -\tau\mathbf{n}, \tag{A.1}$$

where k and τ are the curvature and torsion of the curve, respectively, which is defined by the relation $k = \mathbf{e}_1' \circ \mathbf{e}_2$ and $\tau = \mathbf{e}_2' \circ \mathbf{e}_3$. Now we list some results in [3] on the evolution of the curve in Minkowski space.

Formulation I. We introduce the following complex quantity $\mathbf{N}$ and Hasimoto function ψ,

$$\mathbf{N} = (\mathbf{n} + \mathrm{i}\mathbf{b}) \exp\left(\mathrm{i}\int_{-\infty}^{s} \mathrm{d}s'\tau\right), \tag{A.2}$$

$$\psi = k \exp\left(\mathrm{i} \int_{-\infty}^{s} \mathrm{d}s' \tau \right). \tag{A.3}$$

In terms of $\mathbf{t}$, $\mathbf{N}$ and $\mathbf{N}^*$, the Frenet–Serret equations (A.1) can be rewrite as

$$\mathbf{N}_s = \psi \mathbf{t}, \tag{A.4}$$

$$\mathbf{t}_s = \tfrac{1}{2}(\psi^* \mathbf{N} + \psi \mathbf{N}^*). \tag{A.5}$$

It is easily checked that the new frame $\mathbf{t}$, $\mathbf{N}$ and $\mathbf{N}^*$ satisfies the relations $\mathbf{N} \cdot \mathbf{N}^* = 2$, $\mathbf{N} \cdot \mathbf{t} = \mathbf{N}^* \cdot \mathbf{t} = \mathbf{N} \cdot \mathbf{N} = 0$. The time evolution of $\mathbf{N}$ and $\mathbf{t}$ may be expressed as

$$\mathbf{N}_t = \mathrm{i} R \mathbf{N} + \gamma \mathbf{t}, \tag{A.6}$$

$$\mathbf{t}_t = \tfrac{1}{2}(\gamma^* \mathbf{N} + \gamma \mathbf{N}^*), \tag{A.7}$$

where $R(s, t)$ is real. Using the compatibility condition $\mathbf{N}_{ts} = \mathbf{N}_{st}$, we get the time evolution of the Hasimoto function ψ

$$\psi_t - \gamma_s - \mathrm{i} R \psi = 0, \qquad R_s = \tfrac{\mathrm{i}}{2}(\gamma \psi^* - \gamma^* \psi). \tag{A.8}$$

Formulation II. Taking $\mathbf{L} = \frac{1}{\sqrt{2}}(\mathbf{n} + \mathbf{t}) \exp\left(-\int_{-\infty}^{s} \mathrm{d}s' k\right)$ and $\mathbf{M} = \frac{1}{\sqrt{2}}(\mathbf{n} - \mathbf{t}) \exp\left(\int_{-\infty}^{s} \mathrm{d}s' k\right)$, we may rewritten Frenet–Serret equations (A.1) as

$$\mathbf{b}_s = -\hat{\phi} \mathbf{L} - \hat{\psi} \mathbf{M}, \qquad \mathbf{L}_s = \hat{\psi} \mathbf{b}, \qquad \mathbf{M}_s = \hat{\phi} \mathbf{b}, \tag{A.9}$$

where $\hat{\psi} = \frac{1}{\sqrt{2}} \tau \exp\left(-\int_{-\infty}^{s} k \mathrm{d}s'\right)$ and $\hat{\phi} = \frac{1}{\sqrt{2}} \tau \exp\left(\int_{-\infty}^{s} k \, \mathrm{d}s'\right)$. The time evolution of $\mathbf{b}$, $\mathbf{L}$ and $\mathbf{M}$ can be written as

$$\mathbf{b}_t = \hat{\zeta} \mathbf{L} + \hat{\beta} \mathbf{M}, \qquad \mathbf{L}_t = -\hat{\beta} \mathbf{b} + \hat{\gamma} \mathbf{L}, \qquad \mathbf{M}_t = -\hat{\zeta} \mathbf{b} - \hat{\gamma} \mathbf{M}. \tag{A.10}$$

Using the compatibility conditions $\mathbf{b}_{st} = \mathbf{b}_{ts}$, $\mathbf{L}_{st} = \mathbf{L}_{ts}$ and $\mathbf{M}_{st} = \mathbf{M}_{ts}$, we get

$$\hat{\psi}_t + \hat{\beta}_s - \hat{\gamma} \hat{\psi} = 0, \qquad \hat{\phi}_t + \hat{\zeta}_s + \hat{\gamma} \hat{\phi} = 0, \qquad \hat{\gamma}_s - \hat{\zeta} \hat{\psi} + \hat{\beta} \hat{\phi} = 0. \tag{A.11}$$

Formulation III. Taking $\mathbf{P} = \frac{1}{\sqrt{2}}(\mathbf{b} + \mathbf{t})$ and $\mathbf{Q} = \frac{1}{\sqrt{2}}(\mathbf{b} - \mathbf{t})$, we may rewrite (A.1) as

$$\mathbf{n}_s = \tilde{\psi} \mathbf{P} - \tilde{\phi} \mathbf{Q}, \qquad \mathbf{P}_s = \tilde{\phi} \mathbf{n}, \qquad \mathbf{Q}_s = -\tilde{\psi} \mathbf{n}, \tag{A.12}$$

where $\tilde{\psi} = \frac{1}{\sqrt{2}}(k + \tau)$ and $\tilde{\phi} = \frac{1}{\sqrt{2}}(k - \tau)$. The time evolution of $\mathbf{n}$, $\mathbf{P}$ and $\mathbf{Q}$ may be expressed as

$$\mathbf{n}_t = \tilde{\zeta} \mathbf{P} + \tilde{\beta} \mathbf{Q}, \qquad \mathbf{P}_t = -\tilde{\beta} \mathbf{n} + \tilde{\gamma} \mathbf{P}, \qquad \mathbf{Q}_t = -\tilde{\zeta} \mathbf{n} - \tilde{\gamma} \mathbf{Q}. \tag{A.13}$$

Using the compatibility conditions $\mathbf{n}_{st} = \mathbf{n}_{ts}$, $\mathbf{P}_{st} = \mathbf{P}_{ts}$ and $\mathbf{Q}_{st} = \mathbf{Q}_{ts}$, we get

$$\tilde{\psi}_t - \tilde{\zeta}_s + \tilde{\psi} \tilde{\gamma} = 0, \qquad \tilde{\phi}_t + \tilde{\beta}_s - \tilde{\gamma} \tilde{\phi} = 0, \qquad \tilde{\gamma}_s - \tilde{\beta} \tilde{\psi} - \tilde{\zeta} \tilde{\phi} = 0. \tag{A.14}$$

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